A unified approach for exactly solvable potentials in quantum mechanics using shift operators

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We present a unified approach for solving and classifying exactly solvable potentials. Our unified approach encompasses many well-known exactly solvable potentials. Moreover, the new approach can be used to search systematically for a new class of solvable potentials.

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I. INTRODUCTION

Exactly solvable potentials are extensively applied in the investigation of many physical systems in condensed matter, nuclear physics, quantum optics and solid state physics. Typical examples within this large class of potentials in non-relativistic quantum mechanics include the Morse potential [1], the Pöschl-Teller potential and the Coulomb potential. In particular, the Morse potential has provided a reasonably suitable approximation for studying anharmonicity and bond dissociation of diatomic molecules at low vibrational levels. Recently, coherent states [2], q-deformation [3, 4] and the phase space and transport properties of coupled Morse potentials [5] have also been investigated. The Pöschl-Teller potential on the other hand has always been explored for its short-range properties. Its generalized coherent states [6], nonlinear properties [7, 8] and supersymmetric extension [9] have also been extensively studied. In recent years, under the guise of supersymmetric quantum mechanics, the idea of shaped invariant potentials have also been mooted [14–18]. In particular, a large class of such potential is the Natanzon class [19–22].

Algebraic methods [23, 24] exploiting the underlying Lie symmetry and its associated algebras have been widely used to study many of these exactly solvable potentials, for instance, among many others, Darboux transformation, Infeld-Hull factorization, Mielnick factorization, SUSY quantum mechanics, inverse scattering theory [10] and intertwining technique [9]. Moreover, by exploiting an underlying q-deformed quantum algebraic symmetry, Spiridonov [25] has shown how a finite difference differential equation can generate in the limiting cases a set of exactly solvable potentials. Recently, algebraic method based on nonlinear algebras [4, 7, 8] has also been used to analyze these solvable potentials. The main purpose of this paper is to develop a new unified approach to obtain transition operators for a large class of exactly solvable potentials.

In section II, we briefly sketch the main ideas behind the diagonalization process. We then sketch the main idea behind our method of getting shift operators from Hamiltonians with arbitrary solvable potentials in section III. In section IV, we illustrate our method by showing how most of the exactly solvable potentials can be obtained through the method. Finally in section V, we summarize our technique and discuss some subtle points associated with it.

II. METHOD

Suppose we have a physical system with Hamiltonian H, generalized coordinate operator, Q, and generalized momentum operator, P, satisfying the commutation relations [26]

$$[H, Q] = Q\Theta_1(H, I_i, c_i) + P\Pi_1(H, I_i, c_i), \quad (1a)$$

$$[H, P] = Q\Theta_2(H, I_i, c_i) + P\Pi_2(H, I_i, c_i), \quad (1b)$$

where $I_i,\ i \in \mathcal{S}$ for some arbitrary index set, $\mathcal{S}$, are some invariants of motion of the system and $c_i,\ i \in \mathcal{S}$ are some constants. Denoting

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$$([H,\ Q],\ [H,\ P]) := [H,\ (Q,\ P)], \tag{2}$$

we find that we can recast the previous commutation equations more succinctly as

$$([H,\ Q],\ [H,\ P])U = [H,\ (Q,\ P)U], \tag{3}$$

where U is a 2×2 matrix so that the commutation relations in eq(1) can be rewritten as

$$[H,\ (Q,\ P)] = (Q,\ P)\begin{pmatrix} \Theta_1 & \Theta_2 \\ \Pi_1 & \Pi_2 \end{pmatrix} \equiv (Q,\ P)M. \tag{4}$$

By diagonalizing the matrix, M, on the right hand side of eq(4), we see that

$$[H,\ (Q,\ P)U] = (Q,\ P)UU^{-1}MU$$

giving

$$[H,\ (Q,\ P)U] = (Q,\ P)U\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix},$$

where E_1 and E_2 are the eigenvalues of M. By defining $(Q,\ P)U$ as (S_1, S_2), we see that the operators S_1 and S_2 acts as lowering and raising operators.

In the specific case of the one dimensional harmonic oscillator [11, 13], the Hamiltonian is given by $H = \frac{1}{2}(x^2 + p^2)$ where x is the coordinate operator and $p = -i\frac{d}{dx}$ is the momentum operator with commutator given by $[x,\ p] = i$. Moreover, it is obvious from the Hamiltonian that the following commutation relations hold $[H,\ x] = -ip,\ [H,\ p] = ix$. Identifying Q and P as x and p, we see that

$$[H,\ (x,\ p)] = (x,\ p)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{5}$$

Diagonalizing the matrix on the right hand side of eq(5) gives

$$[H,\ (x,\ p)U] = (x,\ p)UU^{-1}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}U$$

so that

$$[H,\ (x,\ p)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}] = (x,\ p)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus,

$$[H,\ (a,\ a^\dagger)] = (a,\ a^\dagger)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6}$$

and we can therefore identify $a \equiv \frac{1}{\sqrt{2}}(x + ip)$ and $a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip)$ as the natural shift operators arising from the diagonalization process. The harmonic oscillator is the simplest example. In general, it is possible to replace the coordinate operator x and momentum operator p by generalized "coordinate" operator Q and "momentum" operator P which are functions of the operators $x,\ p,\ H$ and $I_i, i \in \mathcal{S}$. Physically, the energy spectrum will no longer be uniformly spaced since the eigenvalues, E_1 and E_2, associated with the raising and lowering operators are in functions of H and the amount through which these operators raise or lower depend very much on the energy level as well. This is the main source of nonlinearity in the algebra. Moreover, we note that the coefficient matrix, M, in eq(4) as well as the corresponding transformation matrix, U, are generally functions of the Hamiltonian H, the invariants of motion I_i and some constants c_i.

III. SHIFT OPERATORS

We next see how the shift operators can be obtained for Hamiltonians with arbitrary solvable potentials. This method can also be regarded as a means of searching systematically for the solvable potentials. To this end, we begin with the following Hamiltonian

$$H = X(x)\frac{d^2}{dx^2} + V(x). \tag{7}$$

where $X(x)$ is an arbitrary function of position and $V(x)$ is an arbitrary potential. Define

$$P(x,p) = Y(x)\frac{d}{dx} + Z(x) \tag{8}$$

with arbitrary functions $Y(x)$ and $Z(x)$ to be determined. It is easy to show that the commutation relation between H and P is

$$\begin{aligned}[H,\ P] = (2X(x)Y'(x) - X'(x)Y(x))\frac{d^2}{dx^2} \\ +X(x)(Y''(x) + 2Z'(x))\frac{d}{dx} + X(x)Z''(x) \\ - Y(x)V'(x).\end{aligned} \tag{9}$$

Since $X(x)$, $Y(x)$ and $Z(x)$ are arbitrary functions, it is possible to set

$$X(x)(Y''(x) + 2Z'(x)) = \alpha Y(x) \tag{10}$$

$$2X(x)Y'(x) - X'(x)Y(x) = (\beta Q(x) + \gamma)X(x), \tag{11}$$

where α, β and γ are arbitrary constants and $Q(x)$ is the generalized "coordinate" to be determined, so that

$$\begin{aligned}[H,\ P] = (\beta Q(x) + \gamma)H + \alpha P - (\beta Q(x) + \gamma)V(x) \\ -\alpha Z(x) - Y(x)V'(x) + X(x)Z''(x).\end{aligned} \tag{12}$$

Furthermore, if we impose for simplicity

$$X(x)Z''(x) - (\beta Q(x) + \gamma)V(x) - \alpha Z(x) - Y(x)V'(x) = Q(x), \quad (13)$$

and solved for the operator $Q(x)$ as

$$Q(x) = \frac{1}{1+\beta V(x)}(X(x)Z''(x) - \gamma V(x) - \alpha Z(x) - Y(x)V'(x)), \quad (14)$$

we find

$$[H,\ P] = Q(x)(\beta H + 1) + \alpha P + \gamma H. \quad (15)$$

Concomitantly, we can also work out explicitly the expression for the commutator of H with $Q(x)$ giving

$$[H,\ Q(x)] = 2X(x)Q'(x)\frac{d}{dx} + X(x)Q''(x). \quad (16)$$

Finally, one can set

$$X(x)Q'(x) = \lambda Y(x), \quad (17)$$

so that

$$[H,\ Q(x)] = 2\lambda P - 2\lambda Z(x) + X(x)Q''(x).$$

If the further constraint

$$-2\lambda Z(x) + X(x)Q''(x) = \nu Q(x) + \tau, \quad (18)$$

is imposed, we arrive at

$$[H,\ Q(x)] = 2\lambda P + \nu Q(x) + \tau. \quad (19)$$

Here, λ, ν and τ are constants to be determined. Note that the quantities τ and γH can be absorbed in certain cases by a redefinition of the operators $Q(x)$ and P so that we finally obtain a semi-closed algebra. In the process, the parameters $\alpha,\ \beta,\ \gamma,\ \lambda,\ \nu,\ \tau$ are not arbitrary but must satisfy some constraints. It is also interesting to note that in the above construction of the shift operators, we are not constrained by commutation relation between $Q(x)$ and P. Moreover, we do not need to require the operators H, Q and P form a closed algebra, although for most of the well known exactly solvable Hamiltonians, these operators always inevitably form a closed algebra. However, to obtain the ground state, we need to require that the Hamiltonian factorizes according to $S_1 S_2 \sim F(H, I_i, c_i)$ for some arbitrary function, F.

In the construction, we should bear in mind that $Y(x),\ Z(x), Q(x)$ and $\alpha,\ \beta,\ \gamma,\ \lambda,\ \nu,\ \tau$ are all determined by $X(x)$ and $V(x)$ when we construct the shift operators for a given Hamiltonian. Conversely, if we proceed with a systematic search for exactly solvable potentials based on this method, we can have as many degrees of freedom as there are free parameters and functions, namely $X(x),\ Y(x),\ Z(x), Q(x)$ and $\alpha,\ \beta,\ \gamma,\ \lambda,\ \nu,\ \tau$.

IV. SOME SOLVABLE POTENTIALS

We can perform a systematic search for a class of exactly solvable potentials by solving the equations (10,11,14,17,18). Furthermore, the method is tantamount to a construction the corresponding shift operators so that it is possible to obtain the energy spectrum and the underlying algebraic formulation if necessary. Indeed, most of well known solvable potentials can be reproduced using the method by an appropriate choice of functions and parameters.

Case 1. $X(x) = -1,\ \ Y(x) = 1.$

On substituting into the equations (10,11,14,17,18), consistency requires the imposition of the following constraints

$$\begin{aligned} Q'(x) &= -\lambda, \\ -\beta Q(x) - \gamma &= 0, \\ -2Z'(x) &= \alpha, \\ -2\lambda Z(x) - Q''(x) &= \nu Q(x) + \tau, \\ V'(x) + (\beta Q(x) + \gamma)V(x) &= -Z''(x) - \alpha Z(x) - Q(x). \end{aligned}$$

It is easy to solve these constrained equations to obtain

$$Q(x) = -\lambda x + c_1,\ \ Z(x) = -\frac{\alpha}{2}x + c_2,$$

$$V(x) = \frac{1}{2}(\lambda + \frac{\alpha^2}{2})x^2 - (\alpha c_2 + c_1)x + c_3$$

and $\beta = 0,\ \gamma = 0,\ \nu = -\alpha$ and $\tau = c_1\nu - 2\lambda c_2$ with the integral constants $c_i, i = 1, 2, 3$. This is the harmonic oscillator.

By substituting the solution of the constrained equations into eq(4), we get

$$\begin{aligned} [H, Q] &= -\alpha Q + 2\lambda P - (2\lambda c_2 + \nu c_1), \\ [H, P] &= Q + \alpha P. \end{aligned}$$

A simple redefinition of the operators Q and P using $Q = \tilde{Q} + f$ and $P = \tilde{P} + g$ where $f = -\alpha\dfrac{2\lambda c_2 + \nu c_1}{\alpha^2 + 2\lambda}$ and $g = \dfrac{2\lambda c_2 + \nu c_1}{\alpha^2 + 2\lambda}$ so that the operators $\tilde{Q}$ and $\tilde{P}$ take the form

$$\begin{aligned} \tilde{Q} &= -\lambda x + c_1 + \alpha\frac{2\lambda c_2 + \nu c_1}{\alpha^2 + 2\lambda} \\ \tilde{P} &= \frac{d}{dx} - \frac{\alpha}{2}x + c_2 - \frac{2\lambda c_2 + \nu c_1}{\alpha^2 + 2\lambda} \end{aligned}$$

immediately give the equations

$$\begin{aligned} [H, \tilde{Q}] &= -\alpha\tilde{Q} + 2\lambda\tilde{P}, \\ [H, \tilde{P}] &= \tilde{Q} + \alpha\tilde{P}, \end{aligned}$$

or more succinctly as

$$[H,(\tilde{Q},\tilde{P})]=(\tilde{Q},\tilde{P})\begin{pmatrix}-\alpha & 1\\ 2\lambda & \alpha\end{pmatrix}$$

which can be easily diagonalized to give the shift operators S_1 and S_2 which satisfies the commutation relation

$$[H,(S_1,S_2)]=(S_1,S_2)\begin{pmatrix}-\sqrt{\alpha^2+2\lambda} & 0\\ 0 & \sqrt{\alpha^2+2\lambda}\end{pmatrix}$$

with

$$S_1=\frac{1}{\alpha-\sqrt{\alpha^2+2\lambda}}\tilde{Q}+\tilde{P}$$
$$S_2=\frac{1}{\alpha+\sqrt{\alpha^2+2\lambda}}\tilde{Q}+\tilde{P}$$

and

$$[S_1,S_2]=-\sqrt{\alpha^2+2\lambda} \tag{20}$$

It is clear from eq(20) that the shift operators S_1 and S_2 are really the usual lowering and raising operators resepctively for the harmonic oscillators (simply set $A=\frac{-S_1}{(\alpha^2+2\lambda)^{1/4}}$ and $A^\dagger=\frac{S_2}{(\alpha^2+2\lambda)^{1/4}}$). Having constructed the shift operators, the ground state eigenvalue and eigenvector can be solved by the action of the lowering operator on a lowest weight state using the equations

$$H\psi_0(x)=E_0\psi_0(x),\quad S_1\psi(x)=0. \tag{21}$$

The results are easily found to be

$$\psi_0=B_3\exp\left(\frac{1}{2}B_1x^2+B_2x\right) \tag{22}$$

$$E_0=\frac{1}{2}\sqrt{\alpha^2+2\lambda}+c_3-\frac{(c_1+c_2\alpha)^2}{\alpha^2+2\lambda}, \tag{23}$$

where $B_i, i=1,2,3$ are constants of integration.

Case 2. $X(x)=-1,\ \ Y(x)=x.$

As in Case 1, a simple substitution into the equations (10,11,14,17,18) requires for consistency the following constraints

$$\begin{aligned}Q'(x)&=-\lambda x,\\ -\beta Q(x)-\gamma&=-2,\\ -2Z'(x)&=\alpha x,\\ -2\lambda Z(x)-Q''(x)&=\nu Q(x)+\tau,\\ xV'(x)+(\beta Q(x)+\gamma)V(x)&=-Z''(x)-\alpha Z(x)-Q(x).\end{aligned}$$

By solving the equations, we obtain

$$Q(x)=-\frac{\lambda}{2}x^2+c_1,\ \ Z(x)=-\frac{\alpha}{4}x^2+c_2,$$

$$V(x)=\frac{1}{8}(\lambda+\frac{\alpha}{2})^2x^2+\frac{c_3}{x^2}+\frac{1}{2}(\frac{\alpha}{2}-\alpha c_2-c_1)$$

with $\beta=0,\ \gamma=2,\ \nu=-\alpha$ and $\tau=\lambda(1-2c_2)+\alpha c_1$. This gives us the radial harmonic oscillator potential. The rest of the details are similar to Case 1, but we will provide some of the crucial results.

The shift operators S_1 and S_2 can be found through a similar method with the explicit form of the operators given by

$$S_1=\tilde{Q}+(\alpha-\sqrt{\alpha^2+2\lambda})\tilde{P}$$
$$S_2=\tilde{Q}+(\alpha+\sqrt{\alpha^2+2\lambda})\tilde{P}$$

as before but with the redefined operators $\tilde{Q}$ and $\tilde{P}$ defined by

$$\begin{aligned}\tilde{Q}&=-\frac{\lambda}{2}x^2+c_1+\frac{4\lambda}{\alpha^2+2\lambda}H\\ &\quad-\frac{\alpha}{\alpha^2+2\lambda}(\lambda+\alpha c_1-2\lambda c_2)\\ \tilde{P}&=x\frac{d}{dx}-\frac{\alpha}{4}x^2+c_2+\frac{2\alpha}{\alpha^2+2\lambda}H\\ &\quad+\frac{1}{\alpha^2+2\lambda}(\lambda+\alpha c_1-2\lambda c_2).\end{aligned}$$

Moreover, the following commutation relation, similar to Case 1, is also satisfied

$$[H,(S_1,S_2)]=(S_1,S_2)\begin{pmatrix}-\sqrt{\alpha^2+2\lambda} & 0\\ 0 & \sqrt{\alpha^2+2\lambda}\end{pmatrix}.$$

Note that these operators depend on H so that the eigenvalues associated with the raising and lowering operators depend on the energy level it is acting on. Thus the tower of states is no longer equally spaced. The commutation relation for the shift operators is

$$[S_1,S_2]=\frac{8\lambda}{\sqrt{\alpha^2+2\lambda}}(2H+c_1+c_2\alpha-\frac{\alpha}{2}). \tag{24}$$

The ground state can be shown to be

$$\psi_0=B_4x^{-(B_2E_0+B_3)}\exp(-\frac{B_1}{2}x^2) \tag{25}$$

with energy given implicitly by

$$(B_2E_0+B_3)^2+(B_2E_0+B_3)-c_3=0. \tag{26}$$

Case 3. $X(x)=-1,\ \ Y(x)=ae^{cx}+be^{-cx}.$

A similar substitution into equations (10,11,14,17,18) and solving the resulting consistency equations give the solution

$$Q(x)=-\frac{\lambda}{c}(ae^{cx}-be^{-cx})+c_1,$$

$$Z(x)=-\frac{\alpha+c^2}{2c}(ae^{cx}+be^{-cx})+c_2,$$

$$V(x) = c_3(ae^{cx} + be^{-cx})^{-2} + \frac{1}{4c^2}[(\alpha + c^2)^2 + 2\lambda]$$

and $\beta = -\frac{2}{\lambda}c^2$, $\gamma = \frac{2c^2}{\lambda}c_1$, $\nu = -\alpha - 2c^2$, $\tau = -2\lambda c_2 - \nu c_1$. This is the generalized Pöschl-Teller potential [12]. Moreover, we also have the the relation $c_1 + \alpha c_2 = 0$.

The shift operators can be found similarly using the technique described in Case 1 and the commutation relations for the shift operators, S_1 and S_2, are

$$[H, S_1] = S_1\left(-c^2 - \sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}\right) \quad (27)$$

$$[H, S_2] = S_2\left(-c^2 + \sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}\right). \quad (28)$$

The explicit form of the shift operators are given by

$$S_1 = -\tilde{Q}\frac{1}{2\lambda}((\alpha + c^2) - \sqrt{(\alpha + c^2)^2 - 4c^2H + 2\lambda)}) + \tilde{P}$$

$$S_2 = -\tilde{Q}\frac{1}{2\lambda}((\alpha + c^2) + \sqrt{(\alpha + c^2)^2 - 4c^2H + 2\lambda)}) + \tilde{P}$$

where

$$\tilde{Q} = Q + \frac{\gamma}{\beta} = Q - c_1,$$
$$\tilde{P} = P - c_2.$$

with commutation relation

$$[\tilde{Q}, \tilde{P}] = \lambda\left(ae^{cx} + be^{-cx}\right)^2,$$

and

$$[S_1, S_2] = -8abc\sqrt{(\frac{\alpha + c^2}{2c})^2 + \frac{\lambda}{2c^2} - H} \quad (29)$$

The ground state is given by

$$\psi_0 = c_4(ae^{cx} + be^{-cx})^{-\sqrt{(\frac{\alpha+c^2}{2c})^2 + \frac{\lambda}{2c^2} - E_0}} \quad (30)$$

and the ground state energy E_0 can be determined through the equation

$$\left((\frac{\alpha + c^2}{2c})^2 + \frac{\lambda}{2c^2} - E_0\right) + c\sqrt{(\frac{\alpha + c^2}{2c})^2 + \frac{\lambda}{2c^2} - E_0} + \frac{c_4}{4ab} = 0. \quad (31)$$

Case 4. $X(x) = -1$, $Y(x) = a\sin(kx) + b\cos(kx)$. For this case, we get

$$Q(x) = \frac{\lambda}{k}(a\cos(kx) - b\sin(kx)) + c_1,$$

$$Z(x) = \frac{\alpha - k^2}{2k}(a\cos(kx) - b\sin(kx)) + c_2,$$

$$V(x) = c_3(a\sin(kx) + b\cos(kx))^{-2} - \frac{(k^2 - \alpha)^2 + 2\lambda}{4k^2}$$

and $\beta = \frac{2k^2}{\lambda}$, $\gamma = -\frac{2}{\lambda}c_1k^2$, $\nu = 2k^2 - \alpha$, $\tau = c_1(\alpha - 2k^2) - 2\lambda c_2$. This is the familiar Pöschl-Teller potential.

The shift operators are given by

$$[H, S_1] = S_1(k^2 - \sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H})$$
$$[H, S_2] = S_2(k^2 + \sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H})$$

with the operators S_1 and S_2 defined by

$$S_1 = (a\sin(kx) + b\cos(kx))\frac{d}{dx} - \frac{1}{2k}(a\cos(kx) - b\sin(kx))\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}$$

$$S_2 = (a\sin(kx) + b\cos(kx))\frac{d}{dx} + \frac{1}{2k}(a\cos(kx) - b\sin(kx))\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}$$

and the commutator given by

$$[S_1, S_2] = -2k(a^2 + b^2)\sqrt{H + \frac{\lambda}{2k^2} + (\frac{\alpha - k^2}{2k})^2}. \quad (32)$$

The ground state, ψ_0, can be solved and found to be given by the formula

$$\psi_0 = c_4(a\sin(kx) + b\cos(kx))^{\frac{1}{2k^2}\sqrt{(\alpha-k^2)^2+2\lambda+4k^2E_0}} \quad (33)$$

where E_0 is the ground state energy determined through the equation

$$\frac{4k^2}{a^2 + b^2}c_3 + 2k^2\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2E_0} - \left((\alpha - k^2)^2 + 2\lambda + 4k^2E_0\right) = 0. \quad (34)$$

Case 5. $X(x) = -x$, $Y(x) = x$.

A similar substitution of above trial functions yields

$$Q(x) = -\lambda x + c_1,\ Z(x) = -\frac{\alpha}{2}x + c_2,$$

$$V(x) = \frac{1}{2}(\lambda + \frac{\alpha^2}{2})x + \frac{c_3}{x} - (c_1 + \alpha c_2)$$

and $\beta = 0$, $\gamma = 1$, $\nu = -\alpha$, $\tau = \alpha c_1 - 2\lambda c_2$. This is the case of Coulomb potential for the hydrogen atom.

The shift operators, S_1 and S_2, satisfy the commutation relations

$$[H, S_1] = -\sqrt{\alpha^2 + 2\lambda}S_1$$
$$[H, S_2] = \sqrt{\alpha^2 + 2\lambda}S_2,$$

in the same manner as the case for the harmonic oscillator in Case 1 and the radial harmonic oscillator in Case 2 but with the operators defined as

$$(S_1, S_2) = (\tilde{Q}, \tilde{P})U$$

where

$$\tilde{Q} = -\lambda x + \frac{2\lambda}{\alpha^2+2\lambda}H + 2\lambda\frac{c_1+\alpha c_2}{\alpha^2+2\lambda},$$
$$\tilde{P} = x\frac{d}{dx} - \frac{\alpha}{2}x + \frac{\alpha}{\alpha^2+2\lambda}H + \alpha\frac{c_1+\alpha c_2}{\alpha^2+2\lambda},$$

and the matrix U given by

$$U = \begin{pmatrix} \frac{1}{\alpha-\sqrt{\alpha^2+2\lambda}} & \frac{1}{\alpha+\sqrt{\alpha^2+2\lambda}} \\ 1 & 1 \end{pmatrix}.$$

It is instructive to note that the commutation relation for the shift operators can be computed succinctly as

$$[S_1, S_2] = -\frac{2}{\sqrt{\alpha^2+2\lambda}}(H + c_1 + \alpha c_2).$$

A direct computation using $S_1\psi_0 = 0$ yields the ground state as

$$\psi_0 = c_4 x^{-\frac{c_1+\alpha c_2+E_0}{\sqrt{\alpha^2+2\lambda}}} e^{\frac{\sqrt{\alpha^2+2\lambda}}{2}x}, \tag{35}$$

where the ground state energy E_0 is given by

$$E_0 = \frac{\sqrt{\alpha^2+2\lambda}}{2}(1 \pm \sqrt{1+4c_3}) - c_1 - \alpha c_2, \tag{36}$$

choosing appropriately the correct minimal value.

Case 6. $X(x) = -e^{cx},\ \ Y(x) = 1.$

As a final case, we look the above choice, so that upon substitution into the appropriate equations gives

$$Q(x) = \frac{\lambda}{c}e^{-cx} + c_1,\ Z(x) = \frac{\alpha}{2c}e^{-cx} + c_2,$$

$$V(x) = \frac{2\lambda+\alpha^2}{4c^2}e^{-cx} + c_3 e^{cx} + \frac{\alpha}{2c}(2c_2 - c) + \frac{c_1}{c}$$

and $\beta = 0,\ \gamma = -c,\ \nu = -\alpha,\ \tau = \alpha c_1 - \lambda(c+2c_2)$. We note that this is simply the case of Morse potential. As in Case 5, the shift operators, S_1 and S_2, obey the relations

$$[H, S_1] = -\sqrt{\alpha^2+2\lambda}S_1$$
$$[H, S_2] = \sqrt{\alpha^2+2\lambda}S_2,$$

with the shift operators defined by

$$(S_1, S_2) = (\tilde{Q}, \tilde{P})U$$

where

$$\tilde{Q} = \frac{\lambda}{c}e^{-cx} - \frac{2\lambda c}{\alpha^2+2\lambda}H,$$
$$\tilde{P} = \frac{d}{dx} + \frac{\alpha}{2c}e^{-cx} - \frac{\alpha c}{\alpha^2+2\lambda}H,$$

and the matrix U given by

$$U = \begin{pmatrix} \frac{1}{\alpha-\sqrt{\alpha^2+2\lambda}} & \frac{1}{\alpha+\sqrt{\alpha^2+2\lambda}} \\ 1 & 1 \end{pmatrix}$$

as before. The commutation relation between the shift operators, S_1 and S_2, now reads

$$[S_1, S_2] = -\frac{2c^2}{\sqrt{\alpha^2+2\lambda}}(H + \frac{2c_1+\alpha(c+2c_2)}{2c}). \tag{37}$$

Moreover, a straightforward calculation yields the ground state as

$$\psi_0 = c_4 \exp\Big\{-\frac{\sqrt{\alpha^2+2\lambda}}{2c^2}e^{-cx} + \frac{\alpha(c+2c_2)+c\sqrt{\alpha^2+2\lambda}+2c_1-2cE_0}{2\sqrt{\alpha^2+2\lambda}}x\Big\}, \tag{38}$$

and the ground state energy, E_0, is determined by

$$E_0 = \frac{1}{2c}\Big\{2c_1 + \alpha(c+2c_2) + c\sqrt{\alpha^2+2\lambda} \pm \sqrt{2c_3(\alpha^2+2\lambda)}\Big\}, \tag{39}$$

with an appropriate choice of sign.

Finally, we would like to comment briefly on the last two cases. Normally, for radial Coulomb problem and Morse oscillator, we deal with the following Schrödinger equations

$$(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2Z}{r} + \frac{Z^2}{n^2})\psi_{v,l}(r) = 0$$

and

$$(-\frac{d^2}{dy^2} + l^2(1-e^{-y})^2 - l^2)\psi_{v,l} = -(l - v - \frac{1}{2})^2\psi_{v,l}$$

respectively. However, it is always possible to transform the above equations into more convenient forms as

$$(-\rho\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho} + \rho)\psi_{v,l} = 2n\psi_{v,l}$$

with $\rho = \frac{Z}{n}r$ and

$$(-e^x\frac{d^2}{dx^2} + (l - v - \frac{1}{2})^2 e^x + e^{-x})\psi_{v,l} = 2l\psi_{v,l}.$$

To get a deeper insight into this transformation, we note that the Hamiltonian concerned can be rewritten as

$$H = -\frac{d^2}{dx^2} + R(x) + \frac{1}{T(x)},$$

where $T(x)$ and $R(x)$ are suitably chosen functions. The eigenvalue problem for stationary eigenstates takes the form

$$(-\frac{d^2}{dx^2} + R(x) + \frac{1}{T(x)})\psi(x) = E\psi(x). \tag{40}$$

Multiplying by the function $T(x)$, we see that the same eigenvalue problem now appear as

$$(-T(x)\frac{d^2}{dx^2} + T(x)(R(x) - E))\psi(x) = -\psi(x). \quad (41)$$

Hence, the original eigenvalue problem is now transformed into an equivalent eigenvalue problem with the Hamiltonian

$$H' = -T(x)\frac{d^2}{dx^2} + T(x)(R(x) - E). \quad (42)$$

It is precisely with the new form that the potential can be solved more easily. For Coulomb and Morse potentials, we have therefore used the modified Hamiltonian.

V. DISCUSSION AND CONCLUSION

It is known that the local behavior of most solvable potentials reduces either to the harmonic oscillator or the Pöschl-Teller potential [27]. This fact is further confirmed by the commutation relations between the Hamiltonian and the shift operators in section IV. A quick glance at the various cases clearly shows that the general form for the commutation relations can be classified into two major categories, those belonging to the harmonic oscillator (Case 1, Case 2, Case 5 and Case 6), corresponding to $\beta = 0$, and those belonging to the Pöschl-Teller potential (Case 3 and Case 4), corresponding to $\beta \neq 0$. Thus, algebraically, we should expect similar behavior for potentials belonging to the same class.

The inherent nonlinearity in the algebraic structure of the shift operators, in which $F_i(H)S_i = S_iG_i(H), i = 1, 2$, should have a deep physical connections to the study of exactly solvable potentials in supersymmetric theory [13, 28]. Moreover, the technique presented can be systematically analyzed to encompass some of these supersymmetric potentials. Indeed, in section IV, we have not entirely exhausted all possibilities and provided a comprehensive list. However, it is in principle possible to do so. By exploring more complicated functions $X(x)$, $Y(x)$ and $Z(x)$, it is expected that more complicated solvable potentials may arise.

Recently, there has been much interest generated regarding the computation of Franck-Condon factors for anharmonic oscillators [29, 30]. In particular the Morse and Pöschl-Teller potentials do not in general provide a simple analytical expression for the Franck-Condon factors [31,32]. Our technique should be able to cast new light on the exact computation of the Franck-Condon factors for such potentials. More work in this direction will be reported elsewhere. Finally, we note that the method can be extended easily to many-body potentials. This work was supported primarily by NUS Research Grant No. RP3982713. In addition, we would also like to thank the referee for his invaluable comments.

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